Journal of Pure and Applied Algebra 215 (2011) 72–76

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

On the Lie algebra of skew-symmetric elements of an enveloping algebra

Salvatore Siciliano

Dipartimento di Matematica "E. De Giorgi", Università del Salento, Via Provinciale Lecce–Arnesano, 73100–Lecce, Italy

ARTICLE INFO

Article history:
Received 2 August 2009
Received in revised form 22 December 2009
Available online 17 March 2010
Communicated by A.V. Geramita

MSC: 16S30; 16W10; 17B60

ABSTRACT

Let L be a restricted Lie algebra over a field of characteristic $p > 2$ and denote by $u(L)$ its restricted enveloping algebra. We establish when the Lie algebra of skew-symmetric elements of $u(L)$ under the principal involution is solvable, nilpotent, or satisfies an Engel condition.

© 2010 Elsevier B.V. All rights reserved.

1. Introduction

Let A be an algebra with involution $*$. Denote by $A^+ := \{x \in A | x^* = x\}$ the set of symmetric elements of A under $*$ and by $A^- := \{x \in A | x^* = -x\}$ the set of skew-symmetric elements. A general question of interest is which properties of A^+ or A^- can be lifted to A (see [11]). For example, a classical result of Amitsur [1] states that if A^+ or A^- satisfies a polynomial identity, then so does A. Furthermore, there is an extensive literature (see e.g.: [4,7,9,10,13–15]) devoted to determining the extent to which the Lie properties of symmetric or skew-symmetric elements of a group algebra FG under the canonical involution (induced from the map $g \mapsto g^{-1},\ g \in G$) determine the Lie properties of the whole group algebra. Recently, similar questions have been investigated for more general involutions of FG as well (see e.g. [5,6,8,12,16]).

Recall that A can be viewed as a Lie algebra via the Lie product defined by $[a, b] = ab - ba$, for every $a, b \in A$. We shall say that A is Lie solvable (respectively, Lie n-Engel or Lie nilpotent) to mean that A is solvable (respectively, n-Engel or nilpotent) as a Lie algebra. Note that the Lie product of A induces a Lie algebra structure on A^-.

Now, let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$ and let $u(L)$ be the restricted enveloping algebra of L. We use the symbol $\top$ for denoting the *principal involution* of $u(L)$, that is, the unique $\mathbb{F}$-antiautomorphism of $u(L)$ such that $x^\top = -x$ for every x in L. We recall that $\top$ is just the antipode of the $\mathbb{F}$-Hopf algebra $u(L)$.

The aim of the present paper is to investigate the Lie properties of the skew-symmetric elements of $u(L)$ with respect to the involution $\top$. For $p > 2$, we determine precisely when the Lie algebra $u(L)^-$ is solvable, nilpotent or n-Engel for some n. An element x of L is p-nilpotent if $x^{[p^m]} = 0$ for some $m \geq 0$; a subset S of L is p-nil if it consists of p-nilpotent element, while it is p-nilpotent if $S^{[p^m]} := \{x^{[p^m]} | x \in S\} = 0$ for some $m \geq 0$. The derived subalgebra of L is denoted by L'. Our main results are as follows.

Theorem 1. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following conditions are equivalent:*

(1) *$u(L)^-$ is solvable;*
(2) *$u(L)$ is Lie solvable;*
(3) *L' is finite-dimensional and p-nilpotent.*

Theorem 2. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following conditions are equivalent:*

(1) *$u(L)^-$ is n-Engel for some n;*
(2) *$u(L)$ is Lie m-Engel for some m;*
(3) *L is nilpotent, L contains a restricted ideal I such that L/I and I' are finite-dimensional, and L' is p-nilpotent.*

E-mail address: salvatore.siciliano@unisalento.it.

0022-4049/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.jpaa.2010.02.033

Theorem 3. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following conditions are equivalent:*

(1) *$u(L)^-$ is nilpotent;*
(2) *$u(L)$ is Lie nilpotent;*
(3) *L is nilpotent and L' is finite-dimensional and p-nilpotent.*

Thus, in odd characteristic, $u(L)^-$ is solvable, n-Engel for some n or nilpotent if and only if so is $u(L)$ as a Lie algebra. We shall show that all the previous results fail in characteristic 2.

Finally, the Lie structure of the skew-symmetric elements of ordinary enveloping algebras $U(L)$ under the principal involution will be discussed. Indeed, for an arbitrary Lie algebra L over a field of characteristic different from 2, we shall prove that $U(L)^-$ is solvable or n-Engel only when L is abelian.

2. Proofs and concluding remarks

The notation used throughout this paper is essentially standard. Let L be a restricted Lie algebra over a field $\mathbb{F}$ of positive characteristic p. We denote by $Z(L)$ the centre of L. We use the symbols $\gamma_i(L)$ $(i \geq 1)$ and $\delta_j(L)$ $(j \geq 0)$ for the terms of the lower central series and the derived series of L, respectively. We adopt the left-normed convention for longer commutators. For a subset S of L we denote by S_p the restricted subalgebra generated by S. Let now $\mathbb{F}\{x_1, x_2, \ldots\}$ be the free algebra over $\mathbb{F}$ and A an $\mathbb{F}$-algebra. Then $0 \neq f(x_1, x_2, \ldots, x_n) \in \mathbb{F}\{x_1, x_2, \ldots\}$ is said to be a polynomial identity for A if $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_1, a_2, \ldots, a_n \in A$. Similarly, let $\mathbb{F}\{x_1, x_1^*, x_2, x_2^*, \ldots\}$ be the free algebra with involution over $\mathbb{F}$ and R an $\mathbb{F}$-algebra with involution τ. Then $0 \neq f(x_1, x_1^*, \ldots, x_n, x_n^*) \in \mathbb{F}\{x_1, x_1^*, x_2, x_2^*, \ldots\}$ is said to be a $*$-polynomial identity for R if $f(a_1, a_1^\tau, \ldots, a_n, a_n^\tau) = 0$ for all $a_1, \ldots, a_n \in R$.

In order to prove Theorems 1–3 we need some preliminary results.

Lemma 1. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Suppose that $u(L)^-$ satisfies a multilinear polynomial identity $f(x_1, x_2, \ldots, x_n) \in \mathbb{F}\{x_1, x_2, \ldots\}$ of degree n. If $Z(L)$ has infinite dimension, then $u(L)$ satisfies f.*

Proof. For every $a \in u(L)$ we have that $a - a^\top \in u(L)^-$, therefore $u(L)$ satisfies the multilinear $*$-polynomial identity

$$g(x_1, x_1^*, \ldots, x_n, x_n^*) := f(x_1 - x_1^*, \ldots, x_n - x_n^*).$$

Note that the $*$-polynomial g is a sum of monomials of degree n involving each x_i or x_i^* but not both. Write

$$g = f_1 + f_2$$

where $f_1 = f(x_1, x_2 - x_2^*, \ldots, x_n - x_n^*)$. Observe that f_1 (respectively f_2) is just the sum of all monomials of g in which x_1 (respectively x_1^*) appears. Thus, for every $z_1 \in Z(L)$ and $a_1, a_2, \ldots, a_n \in u(L)$, we have

$$\begin{aligned} 0 &= g(z_1a_1, (z_1a_1)^\top, a_2, a_2^\top, \ldots, a_n, a_n^\top) \\ &= z_1f_1(a_1, a_1^\top, \ldots, a_n, a_n^\top) + z_1^\top f_2(a_1, a_1^\top, \ldots, a_n, a_n^\top). \end{aligned}$$

Since one also has $z_1^\top g = z_1^\top f_1 + z_1^\top f_2$, it follows that $(z_1 - z_1^\top)f_1 = 2z_1f_1$ vanishes on $u(L)$. As $p \neq 2$ we get that z_1f_1 vanishes on $u(L)$. A repeated application of this argument allows us to conclude that $z_1z_2 \cdots z_nf(x_1, x_2, \ldots, x_n)$ vanishes on $u(L)$ for every $z_1, z_2, \ldots, z_n \in Z(L)$.

Now suppose, by contradiction, that for some $h_1, h_2, \ldots, h_n \in u(L)$ one has

$$h := f(h_1, h_2, \ldots, h_n) \neq 0.$$

By assumption, there is an infinite set $\mathfrak{Z}$ of $\mathbb{F}$-linearly independent elements of $Z(L)$. Extend $\mathfrak{Z}$ in order to form an ordered $\mathbb{F}$-basis $\mathfrak{B}$ of L. Then, by the PBW Theorem for restricted Lie algebras (see [23], Chapter 2, Theorem 5.1), there exist $y_1, y_2, \ldots, y_m \in \mathfrak{B}$ such that h is an $\mathbb{F}$-linear combination of the elements $y_1^{r_1} \cdots y_m^{r_m}$ where $0 \leq r_i \leq p-1$ for every $i = 1, 2, \ldots, m$. Let $z_1, z_2, \ldots, z_n \in \mathfrak{Z} \backslash \{y_1, y_2, \ldots, y_m\}$. Then the PBW Theorem implies that

$$z_1z_2 \cdots z_nh \neq 0,$$

contradicting what was proved above. Therefore f is a polynomial identity for $u(L)$, as required. □

Corollary 1. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$ such that $Z(L)$ has infinite dimension. If $u(L)^-$ is solvable (respectively nilpotent) then $u(L)$ is Lie solvable (Lie nilpotent) with the same derived length (nilpotency class).*

Proof. Consider the sequence of elements in the free algebra $\mathbb{F}\{x_1, x_2, \ldots\}$ defined inductively by setting $f_1(x_1, x_2) = [x_1, x_2]$ and, for every $n > 1$, $f_n(x_1, x_2, \ldots, x_{2^n}) = [f_{n-1}(x_1, \ldots, x_{2^{n-1}}), f_{n-1}(x_{2^{n-1}+1}, \ldots, x_{2^n})]$. Let m denote the derived length of $u(L)^-$, so that $u(L)^-$ satisfies the multilinear polynomial identity f_m. By Lemma 1 we conclude that $u(L)$ satisfies f_m, and the claim for Lie solvability follows. The proof for Lie nilpotency is analogous. □

Lemma 2. *Let L be a restricted Lie algebra over a field of characteristic $p > 2$. If $u(L)^-$ is solvable then L' is p-nil.*

Proof. Suppose, if possible, that there exist $x, y \in L$ such that the commutator $[x, y]$ is not p-nilpotent. Put $H := \langle x, y \rangle_p$ and let J denote the Jacobson radical of $u(H)$. Then J is $\top$-invariant and $(u(H)/J)^-$ under the induced involution is solvable. Since $u(H)/J$ is semiprimitive, by Proposition 2.4 of [15] we conclude that the elements of $(u(H)/J)^-$ commute and so, in particular, $[x, y] \in J$. Moreover, as $u(H)$ is finitely generated as an associative algebra and, in view of Amitsur's Theorem [1], it satisfies a polynomial identity, by the Razmyslov–Kemer–Braun Theorem (see [3]) its Jacobson radical is nilpotent, and thus $[x, y]$ is p-nilpotent, a contradiction. Since L is solvable and all its commutators are p-nilpotent it follows at once that L' is p-nil, completing the proof. □

Remark 1. Let L be a restricted Lie algebra over a field of characteristic $p > 2$. If I is a restricted ideal of L and A denotes the associative ideal of $u(L)$ generated by I, then one has $A = Iu(L) = u(L)I$. This implies, in particular, that A is $\top$-invariant and the Lie algebra $(u(L)/A)^-$ under the induced involution is a homomorphic image of $u(L)^-$.

We recall that the FC–centre of a restricted Lie algebra L is defined as $\Delta(L) := \{x \in L| \dim_{\mathbb{F}}[L, x] < \infty\}$. It is immediate to see that $\Delta(L)$ is a restricted ideal of L (cf. [17] or [18]).

Proof of Theorem 1. By Theorem 1.3 of [19], the conditions (2) and (3) are equivalent. Moreover, (2) obviously implies (1). Suppose then that $u(L)^-$ is solvable. By Lemma 2 we already know that L' is p-nil. Moreover, by Amitsur's Theorem, $u(L)$ satisfies a polynomial identity and then, by Theorem 6.1 and Lemma 6.2 of [17], $\Delta(L)'$ is finite-dimensional. Thus, in order to complete the proof, it is enough to show that $L = \Delta(L)$. For this purpose, we assume the contrary and proceed to derive a contradiction.

We first observe that we may assume, without loss of generality, that $\Delta(L)$ is abelian. Indeed, put $\mathfrak{L} := L/\Delta(L)'_p$ and note that, since $\Delta(L)'_p$ is finite-dimensional, from Lemma 2.7 of [19] it follows that $\Delta(\mathfrak{L})$ is abelian and different from $\mathfrak{L}$. Also, as $u(\mathfrak{L}) \cong u(L)/\Delta(L)'_p u(L)$, by Remark 1 the Lie algebra $u(\mathfrak{L})^-$ is solvable. Then replace, if necessary, L by $\mathfrak{L}$.

Moreover, as L is solvable (because its elements are skew-symmetric), there is a minimal $i > 0$ such that $\delta_i(L) \subseteq \Delta(L)$. Put $\hat{H} := \delta_{i-1}(L)_p + \Delta(L)$. By Proposition 5.2 of [17], $\Delta(L)$ has finite codimension in L, thus by Lemma 2.7 of [19] we see that $\Delta(\hat{H}) = \Delta(L) \subset \hat{H}$. Also, $\hat{H}/\Delta(\hat{H})$ is abelian and $u(\hat{H})^-$ solvable. Therefore we can replace, if necessary, L by $\hat{H}$ and assume that $L/\Delta(L)$ is abelian. Again, by the corollary after Lemma 5.1.1 of [11] we can suppose, without loss of generality, that the ground field is algebraically closed. Thus, by Proposition 3.6 in Chapter 2 of [23] (applied to $L/\Delta(L)$) we can find an element $x \notin \Delta(L)$ such that either $x^{[p]}$ or $x^{[p]} - x$ is in $\Delta(L)$. Consider the restricted subalgebra

$$H := Fx + \Delta(L). \tag{1}$$

Then $u(H)^-$ is solvable and, by Lemma 2.7 of [19], $\Delta(H) = \Delta(L) \neq H$. Moreover, we have on H that either $(\mathrm{ad}x)^p = 0$ or $(\mathrm{ad}x)^p = \mathrm{ad}x$. Let us proceed by considering these two cases separately.

Suppose first $(\mathrm{ad}x)^p = 0$. In this case H is nilpotent of class at most p and then, as H' has infinite dimension, by [23] (Chapter 2, Proposition 1.3) there exists $2 \le i \le p$ such that

$$\dim_{\mathbb{F}} \gamma_i(H)_p/\gamma_{i+1}(H)_p = \infty. \tag{2}$$

Consider the restricted Lie algebra

$$\tilde{H} := \frac{Fx + Z(H) + \gamma_{i-1}(H)_p}{\gamma_{i+1}(H)_p}.$$

Then $\tilde{H}$ is nilpotent of class 2 and so one has

$$\tilde{H}' = \frac{\gamma_i(H) + \gamma_{i+1}(H)_p}{\gamma_{i+1}(H)_p} \subseteq Z(\tilde{H}).$$

Since $\gamma_i(H)$ is p-nil, the previous relation and (2) imply that both $\tilde{H}'$ and $Z(\tilde{H})$ have infinite dimension. Consequently, as $u(\tilde{H})^-$ is solvable, Corollary 1 allows us to conclude that $u(\tilde{H})$ is Lie solvable. Thus, by Theorem 1.3 of [19], $\tilde{H}'$ must be finite-dimensional, a contradiction.

Now suppose that $(\mathrm{ad}x)^p = \mathrm{ad}x$ on H. By (1), for every $a \in H'$ there exists $b \in \Delta(H)$ such that $a = [b, x]$. It follows that $a(\mathrm{ad}x)^{p-1} = b(\mathrm{ad}x)^p = b\mathrm{ad}x = a$, hence $(\mathrm{ad}x)^{p-1}$ agrees with the identity on H'. As a consequence, one has

$$H' = \bigoplus_{\lambda \in \mathbb{F}_p \setminus \{0\}} V_\lambda, \tag{3}$$

where $\mathbb{F}_p$ is the prime subfield of $\mathbb{F}$ and V_λ denotes the eigenspace of H' relative to the eigenvalue λ for $\mathrm{ad}x$. Now, denote by $E(H')$ and $E(u(\Delta(H)))$ the set of eigenvectors for $\mathrm{ad}x$ in H' and $u(\Delta(H))$, respectively. Also, for $a \in E(u(\Delta(H)))$ we write λ_a for its eigenvalue. Note that, if $a, b \in E(u(\Delta(H)))$ then $(ab)\mathrm{ad}x = [a, x]b + a[b, x] = (\lambda_a + \lambda_b)ab$, so that $ab \in E(u(\Delta(H)))$ and $\lambda_{ab} = \lambda_a + \lambda_b$.

We claim that, for all non-negative integers r, n with $r \ge 2^n - 1$, if $a_1, a_2, \ldots, a_r \in E(H')$ then there exists $\xi \in u(\Delta(H))$ such that the element $xa_1^2a_2^2 \cdots a_r^2 + \xi$ is in $\delta_n(u(H)^-)$. We proceed by induction on n. The claim is true for $n = 0$. In fact, for every $a_1, \ldots, a_r \in E(H')$ we have

$$2xa_1^2 \cdots a_r^2 - [x, a_1^2 \cdots a_r^2] = xa_1^2 \cdots a_r^2 - (xa_1^2 \cdots a_r^2)^\top \in u(H)^-.$$

Since $p > 2$, for $\xi = -\frac{1}{2}[x, a_1^2 \cdots a_r^2] \in u(\Delta(H))$ we get the claim. Thus assume $n > 0$ and let $a_1, a_2, \ldots, a_r \in E(H')$ with $r \ge 2^n - 1$. For a subset C of $\{1, 2, \ldots, r\}$ we put $\lambda_C := \sum_{i \in C} \lambda_{a_i}$ and $a_C := \prod_{i \in C} a_i$, the order of multiplication here being irrelevant as H' is abelian.

We want to show that there is a subset S of $\{1, 2, \dots, r\}$ such that S and its complement $\bar{S}$ in $\{1, 2, \dots, r\}$ have both cardinality at least $2^{n-1} - 1$ and $\lambda_S \neq \lambda_{\bar{S}}$. Assume, if possible, the contrary and put $\lambda := \sum_{i=1}^{r} \lambda_{a_i}$. Then for every $T \subset \{1, 2, \dots, r\}$ with $|T| = 2^{n-1} - 1$ or $|T| = 2^{n-1}$ we have $\lambda_T = \lambda_{\bar{T}} = \frac{\lambda}{2}$. In particular, it follows that

$$\sum_{i=1}^{2^{n-1}-1} \lambda_{a_i} = \sum_{i=1}^{2^{n-1}} \lambda_{a_i} = \frac{\lambda}{2},$$

which forces $\lambda_{a_{2^{n-1}}} = 0$, a contradiction to (3). Therefore there exists a subset S of $\{1, 2, \dots, r\}$ with the required properties.

Now, by the inductive assumption there are ξ_S and $\xi_{\bar{S}}$ in $u(\Delta(H))$ such that $xa_S^2 + \xi_S$ and $xa_{\bar{S}}^2 + \xi_{\bar{S}}$ are in $\delta_{n-1}(u(H)^-)$. Using the fact that $\Delta(H)$ is an abelian ideal of H, by standard calculations we obtain

$$2(\lambda_{a_S} - \lambda_{a_{\bar{S}}})xa_S^2 a_{\bar{S}}^2 + \zeta = [xa_S^2 + \xi_S, xa_{\bar{S}}^2 + \xi_{\bar{S}}] \in \delta_n(u(H)^-),$$

where we put $\zeta := [x, \xi_{\bar{S}}]a_S^2 + [\xi_S, x]a_{\bar{S}}^2 \in u(\Delta(H))$. Since $\lambda_{a_S} = \lambda_S \neq \lambda_{\bar{S}} = \lambda_{a_{\bar{S}}}$ and $p > 2$, by setting $\xi := (2(\lambda_{a_S} - \lambda_{a_{\bar{S}}}))^{-1}\zeta \in u(\Delta(H))$ we get

$$xa_1^2 a_2^2 \cdots a_r^2 + \xi = xa_S^2 a_{\bar{S}}^2 + \xi \in \delta_n(u(H)^-),$$

completing the inductive step.

Finally, since $\dim_{\mathbb{F}} H' = \infty$, in view of (3) for every $m \geq 0$ we can find $\mathbb{F}$-linearly independent elements $a_1, a_2, \dots, a_{2^m-1}$ of $E(H')$. Thus, for the proved claim there exists $\xi \in u(\Delta(H))$ such that the element $y := xa_1^2 a_2^2 \cdots a_{2^m-1}^2 + \xi$ is in $\delta_m(u(H)^-)$. Moreover, since $p > 2$ the PBW Theorem ensures that $y \neq 0$, so that $\delta_m(u(H)^-) \neq 0$ for every $m \geq 0$, contradicting the solvability of $u(H)^-$. □

The following example shows that Theorem 1 fails in characteristic 2.

Example 1. Let L be the restricted Lie algebra over a field $\mathbb{F}$ of characteristic 2 with a basis $\{x_1, x_2, y, z\}$ such that $[x_1, x_2] = z$, $[x_1, y] = x_1$, $[x_2, y] = x_2$, $z \in Z(L)$, $x_1^{[2]} = x_2^{[2]} = 0$, $y^{[2]} = y$ and $z^{[p]} = z$. It is easy to see that $u(L)^-$ coincides with the $\mathbb{F}$-vector space spanned by the set $\{1, x_1, x_2, y, z, x_1z, x_2z, yz, x_1x_2 + x_1x_2z, x_1x_2y + x_1x_2yz\}$. Consequently, $u(L)^-$ is solvable. On the other hand, since the element $yz + x_1x_2 = [[[x_1, y], [x_1, x_1y]], x_2]$ is not nilpotent, from [21] it follows that the algebra $u(L)$ is not Lie solvable.

Proof of Theorem 2. By Theorem 1.2 of [19] the conditions (2) and (3) are equivalent and, clearly, (2) implies (1). Assume now that condition (1) holds. Since the elements of L are skew-symmetric it is clear that L is m-Engel. Moreover, by Amitsur's Theorem $u(L)$ satisfies a polynomial identity and then, by Corollary 2.2 of [19], L contains a nilpotent restricted ideal I such that L/I and I' are finite-dimensional. Clearly, without loss of generality, we can suppose that $Z(L) \subseteq I$. Consequently, for any t with $p^t \geq m$ we have $L^{[p^t]} \subseteq Z(L) \subseteq I$, so that L/I is finite-dimensional and p-nilpotent. Therefore, Proposition 5.1 in [20] allows us to conclude that L is nilpotent.

It remains only to show that L' is p-nilpotent. Suppose, if possible, the contrary. Then, since L is nilpotent, there exists a minimal $n > 2$ such that $\gamma_n(L)$ is p-nilpotent. By [23] (Chapter 2, Proposition 1.3), $\gamma_n(L)_p$ is a p-nilpotent restricted ideal of L. Put $\mathfrak{L} := L/\gamma_n(L)_p$. Since $\gamma_{n-1}(\mathfrak{L}) \subseteq Z(\mathfrak{L})$, from the assumption it follows that there exist $x, y \in \mathfrak{L}$ such that $z := [x, y] \in Z(\mathfrak{L})$ and $z^{[p^t]} \neq 0$ for every $t \geq 0$. Clearly, $2xyz - z^2$ is a skew-symmetric element of $u(\mathfrak{L})$. Moreover, it can be seen by an easy induction that for every $r > 0$ one has

$$[x, \underbrace{2xyz - z^2, \dots, 2xyz - z^2}_{r \text{ times}}] = 2^r xz^{2r}. \tag{4}$$

Now, by Remark 1 the associative ideal J generated by $\gamma_n(L)_p$ is $\top$-invariant and the Lie algebra $(u(L)/J)^-$ under the induced involution is m-Engel. Since $u(L)/J \cong u(\mathfrak{L})$ and $p \neq 2$, from (4) we deduce that $xz^{p^t} = 0$ for every t with $p^t \geq 2m$. On the other hand, since x and $z^{[p^t]}$ are $\mathbb{F}$-linearly independent (as $z^{[p^t]}$ is central whereas x is not), the last conclusion contradicts the PBW Theorem, completing the proof. □

Proof of Theorem 3. Obviously, if $u(L)$ is Lie nilpotent then $u(L)^-$ is nilpotent. On the other hand, if $u(L)^-$ is nilpotent then, as the elements of L are skew-symmetric, L is clearly nilpotent. Furthermore, by Theorem 1 we infer that L' is finite-dimensional and p-nilpotent. At this stage, Theorem 1.1 of [19] allows us to conclude that $u(L)$ is Lie nilpotent, and the proof is complete. □

Note that Theorems 2 and 3 do not hold in characteristic 2, as the following example shows.

Example 2. Let L be the restricted Lie algebra over a field $\mathbb{F}$ of characteristic 2 with a basis $\{x, y, z\}$ such that $[x, y] = z$, $z \in Z(L)$, $x^{[2]} = y^{[2]} = 0$ and $z^{[2]} = z$. Then it is immediate to see that $u(L)^-$ coincides with the $\mathbb{F}$-vector space spanned by the set $\{1, x, y, z, xz, yz, xy + xyz\}$. As a consequence, $u(L)$ is nilpotent. On the other hand, by Theorem 1.2 of [19], $u(L)$ cannot satisfy any Engel condition.

Finally, for the Lie algebra of skew-symmetric elements of ordinary enveloping algebras under the principal involution we have the following

Corollary 2. *Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic $p \neq 2$. Then $U(L)^-$ is solvable or n-Engel for some n if and only if L is abelian.*

Proof. The condition is obviously sufficient. Let us prove the converse. If the ground field $\mathbb{F}$ has characteristic zero then, since $U(L)$ satisfies a polynomial identity by Amitsur's Theorem, in view of [2] (Theorem 25 in Section 6.7) L is necessarily abelian. Suppose then $p > 2$. Put

$$\hat{L} := \sum_{k \geq 0} L^{p^k} \subseteq U(L)$$

where L^{p^k} is the $\mathbb{F}$-vector space spanned by the set $\{l^{p^k} \mid l \in L\}$. Then $\hat{L}$ is a restricted Lie algebra with $h^{[p]} = h^p$ for all $h \in \hat{L}$. Moreover, by Corollary 1.1.4 of [22] we have $U(L) = u(\hat{L})$. Therefore, as $u(\hat{L})^-$ is solvable or n-Engel, Theorems 1 and 2 imply that $\hat{L}'$ is p-nilpotent. Since $u(\hat{L}) = U(L)$ is a domain, the last condition occurs only when $L' = 0$, and the claim follows. $\square$

Acknowledgement

I am grateful to the anonymous referee for the useful comments.

References

[1] S.A. Amitsur, Identities in rings with involutions, Israel J. Math. 7 (1969) 63–68.
[2] Yu.A. Bahturin, Identical Relations in Lie Algebras, VNU Science Press, Utrecht, 1987.
[3] A. Braun, The nilpotency of the radical in a finitely generated PI ring, J. Algebra 89 (1984) 375–396.
[4] O. Broche Christo, C. Polcino Milies, Commutativity of skew symmetric elements in group rings, Proc. Edinb. Math. Soc. (2) 50 (2007) 37–47.
[5] O. Broche Christo, E. Jespers, M. Ruiz, Antisymmetric elements in group rings with an orientation morphism, Forum Math. 21 (2009) 427–454.
[6] O. Broche Christo, E. Jespers, C. Polcino Milies, M. Ruiz, Antisymmetric elements in group rings. II, J. Algebra Appl. 8 (2009) 115–127.
[7] A. Giambruno, C. Polcino Milies, Unitary units and skew elements in group algebras, Manuscripta Math. 111 (2003) 195–209.
[8] A. Giambruno, C. Polcino Milies, S.K. Sehgal, Lie properties of symmetric elements in group rings, J. Algebra 321 (2009) 890–902.
[9] A. Giambruno, S.K. Sehgal, Lie nilpotence of group rings, Comm. Algebra 21 (1993) 4253–4261.
[10] A. Giambruno, S.K. Sehgal, Group algebras whose Lie algebra of skew-symmetric elements is nilpotent, in: Group, Rings and Algebras, in: Contemp. Math, vol. 420, Amer. Math. Soc, Providence, RI, 2006, pp. 113–120.
[11] I.N. Herstein, Rings with Involution, University of Chicago Press, Chicago, 1976.
[12] E. Jespers, M. Ruiz, Antisymmetric elements in group rings, J. Algebra Appl. 4 (2005) 341–353.
[13] G. Lee, Group rings whose symmetric elements are Lie nilpotent, Proc. Amer. Math. Soc. 127 (1999) 3153–3159.
[14] G. Lee, The Lie n-Engel property in group rings, Comm. Algebra 28 (2000) 867–881.
[15] G. Lee, S.K. Sehgal, E. Spinelli, Group algebras whose symmetric and skew elements are Lie solvable, Forum Math. 21 (2009) 661–671.
[16] G. Lee, S.K. Sehgal, E. Spinelli, Lie properties of symmetric elements in group rings. II, J. Pure Appl. Algebra 213 (2009) 1173–1178.
[17] D.S. Passman, Enveloping algebras satisfying a polynomial identity, J. Algebra 134 (1990) 469–490.
[18] V.M. Petrogradsky, Existence of identities in the restricted enveloping algebra, Math. Zametki 49 (1991) 84–93.
[19] D.M. Riley, A. Shalev, The Lie structure of enveloping algebras, J. Algebra 162 (1993) 46–61.
[20] A. Shalev, Polynomial identities in graded group rings, restricted Lie algebras and p-adic analytic groups, Trans. Amer. Math. Soc. 337 (1993) 451–462.
[21] M.B. Smirnov, A.E. Zalesski, Associative rings satisfying the identity of Lie solvability, Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk 123 (1982) 15–20.
[22] H. Strade, Simple Lie Algebras over Fields of Positive Characteristic I, Structure Theory, Walter de Gruyter & Co, Berlin, New York, 2004.
[23] H. Strade, R. Farnsteiner, Modular Lie Algebras and their Representations, Marcel Dekker, New York, 1988.